# A Remark on Asymptotics of the Gamma Function at Infinity (Study on Applications for Fractional Calculus Operators in Univalent Function Theory)



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**Abstract**

The asymptotic estimate of the gamma function $|\Gamma(x+iy)|$ when $|y|$ tends to infinity is proved. The result gives a more precise one of the known formula.

**KEY WORDS**: gamma function, asymptotic estimate
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Let $\Gamma(z)$ be the gamma function of complex argument $z=x+iy$ $(x,y\in\mathbb{R}=(-\infty,\infty))$ (see [1, Chapter 1]). It is known [1, 1.18(2)] the following asymptotic formula of $\Gamma(z)$ at infinity:

$$\Gamma(z)=\sqrt{2\pi}e^{-z}e^{(z-1/2)\log z}\left[1+O\left(\frac{1}{z}\right)\right] \quad (|z|\to\infty,\ |\arg(z)|<\pi), \tag{1}$$

which is called Stirling formula. As a corollary of this result, in [1, 1.18(6)] the asymptotic relation for $|\Gamma(x+iy)|$, as $|y|\to\infty$, is presented in the form

$$\lim_{|y|\to\infty}|\Gamma(x+iy)|e^{|y|\pi/2}|y|^{1/2-x}=\sqrt{2\pi}. \tag{2}$$

This relation needs in correction, namely the multiplier $e^{-x}$ must be added:

$$\lim_{|y|\to\infty}|\Gamma(x+iy)|e^{|y|\pi/2}|y|^{1/2-x}=\sqrt{2\pi}e^{-x}. \tag{3}$$

This result follows from the following assertion.

**Theorem.** *For fixed $x \in \mathbb{R}$ there holds the following asymptotic estimate for $|\Gamma(x+iy)|$ at infinity:*

$$|\Gamma(x+iy)| = \sqrt{2\pi}|y|^{x-1/2}e^{-x-|y|\pi/2}\left[1+O\left(\frac{1}{|y|}\right)\right] \quad (|y|\to\infty). \tag{4}$$

**Proof.** Put $z = x+iy \in \mathbb{C}$ and fix the real part $x$. From (1) we have, as $|y| \to \infty$,

$$|\Gamma(x+iy)| = (2\pi)^{1/2}\left|e^{-x-iy}e^{(x-1/2+iy)[\log|x+iy|+i\arg(x+iy)]}\right|\left[1+O\left(\frac{1}{x+iy}\right)\right]$$

or

$$|\Gamma(x+iy)| = (2\pi)^{1/2}e^{-x}e^{(x-1/2)\log|x+iy|}e^{-y\arg(x+iy)}\left[1+O\left(\frac{1}{x+iy}\right)\right]. \tag{5}$$

Estimating more precisely the terms in (5), when $|y| \to \infty$, we have

$$\begin{aligned} e^{(x-1/2)\log|x+iy|} &= e^{(x-1/2)\log|y|}e^{\{(x-1/2)/2\}\log(1+x^2/y^2)} \\ &= |y|^{x-1/2}\left(1+\frac{|x|^2}{|y|^2}\right)^{(x-1/2)/2} \end{aligned}$$

and the last relation means

$$e^{(x-1/2)\log|x+iy|} = |y|^{x-1/2}\left[1+O\left(\frac{1}{|y|^2}\right)\right] \quad (|y|\to\infty). \tag{6}$$

Further, since $\arg(x+iy) \to \pi/2$ as $y \to \infty$ and $\arg(x+iy) \to -\pi/2$ as $y \to -\infty$, the term $e^{-y\arg(x+iy)}$ can be represented as

$$e^{-y\arg(x+iy)} = e^{-|y|\pi/2}e^{-|y|[\arg(x+iy)\mathrm{sign}(y)-\pi/2]}.$$

It follows from the l'Hospital rule that

$$e^{-|y|[\arg(x+iy)\mathrm{sign}(y)-\pi/2]} = 1+O\left(\frac{1}{|y|}\right) \quad (|y|\to\infty),$$

and hence

$$e^{-y\arg(x+iy)} = e^{-|y|\pi/2}\left[1+O\left(\frac{1}{|y|}\right)\right] \quad (|y|\to\infty). \tag{7}$$

Finally, if $|y| \to \infty$, then $|x+iy| \sim |y|$, and hence

$$O\left(\frac{1}{x+iy}\right) = O\left(\frac{1}{|y|}\right) \quad (|y|\to\infty). \tag{8}$$

Substituting (6), (7) and (8) into (5), we obtain (4). This completes the proof of theorem.

**Corollary 1.** *For fixed $x \in \mathbb{R}$ there holds the asymptotic estimate* (3) *for* $|\Gamma(x+iy)|$ *at infinity, which is equivalent to the relation*

$$|\Gamma(x+iy)| \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-x-|y|\pi/2} \quad (|y| \to \infty). \tag{9}$$

**Remark 1.** The estimate (9) was indicated in our paper [2, the relation (2.2)].

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Taking $x = -k$ $(k \in \mathbb{N}_0)$ in Theorem, we obtain the following assertion.

**Corollary 2.** *If $k \in \mathbb{N}_0$ is fixed, then there holds the estimate*

$$|\Gamma(-k+iy)| = (2\pi)^{1/2}|y|^{-k-1/2}e^{k-|y|\pi/2}\left[1+O\left(\frac{1}{|y|}\right)\right] \quad (|y| \to \infty). \tag{10}$$

*In particular, for $k = 0$*

$$|\Gamma(iy)| = (2\pi)^{1/2}|y|^{-1/2}e^{-|y|\pi/2}\left[1+O\left(\frac{1}{|y|}\right)\right] \quad (|y| \to \infty). \tag{11}$$

**Corollary 3.** *If $k \in \mathbb{N}_0$ is fixed and $n \in \mathbb{N}$, then there holds the estimate*

$$|\Gamma(-k \pm in)| = (2\pi)^{1/2}n^{-k-1/2}e^{k-n\pi/2}\left[1+O\left(\frac{1}{n}\right)\right] \quad (n \to \infty). \tag{12}$$

*In particular, for $k = 0$*

$$|\Gamma(\pm in)| \sim (2\pi)^{1/2}n^{-1/2}e^{-n\pi/2}\left[1+O\left(\frac{1}{n}\right)\right] \quad (n \to \infty). \tag{13}$$

**Remark 2.** It is known (see, for example, [3, (3.39)]) that the gamma function has the following asymptotic near the poles $z = -k$ $(k \in \mathbb{N}_0)$

$$\Gamma(z) = \frac{(-1)^k}{k!(z+k)}\left[1+O\left(\frac{1}{z}\right)\right] \quad (z \to -k,\ k \in \mathbb{N}_0). \tag{14}$$

The relation (10) yields the asymptotic estimate of $|\Gamma(z)|$ at infinity on the line parallel to the imaginary axis $z = -k + iy$ $(k \in \mathbb{N}_0)$ and, in particular, (11) gives the asymptotic estimate of $|\Gamma(z)|$ on the imaginary axis $z = iy$.

**Remark 3.** It is well known that

$$\Gamma(n) = (n-1)! \quad (n \in \mathbb{N}_0). \tag{15}$$

Then (14) and (15) yield different asymptotic behaviors of $|\Gamma(z)|$ as $z \to -n$ and $z \to n$ $(n \in \mathbb{N})$ at the positive and negative integers:

$$|\Gamma(z)| \sim \frac{1}{n!|z+n|} \quad (z \to -n;\ n \in \mathbb{N}); \qquad |\Gamma(z)| \sim (n-1)! \quad (z \to n;\ n \in \mathbb{N}). \tag{16}$$

On the other hand, (13) gives the same asymptotic estimates for $|\Gamma(z)|$ as $z \to -in$ and $z \to in$ $(n \in \mathbb{N})$ on the imaginary axis

$$|\Gamma(z)| \sim (2\pi)^{1/2} n^{-1/2} e^{-n\pi/2} \quad (z \to \pm in;\ n \in \mathbb{N}). \tag{17}$$

Moreover, the relation (9) show that $|\Gamma(x+iy)|$ for fixed $x \in \mathbb{R}$ has the same forms as $y \to -\infty$ and $y \to +\infty$

$$|\Gamma(x+iy)| \sim (2\pi)^{1/2} |y|^{x-1/2} e^{-x-|y|\pi/2} \quad (y \to \pm\infty). \tag{18}$$

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